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O(d,d;R) Deformations of Complex Structures and Extended Worldsheet Supersymmetry

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ABSTRACT

It is shown that the $O(d,d;\mathrm{R})$ deformations of the superstring vacua and the $O(d,d+16;\mathrm{R})$ deformations of the heterotic string vacua preserve extended worldsheet supersymmetry and, hence, generate superconformal deformations. The transformations of the complex structures are given explicitly and the action of the discrete duality subgroup is discussed. The results are valid when the complex structures are independent of the d coordinates which appear in the transformations. It is shown that generic deformations do not preserve the known superfield formulations of (2,2) extended supersymmetry. The analysis is performed by decomposing the transformations in terms of the metric vielbein and by introducing space-time connections induced due to the non-linear action of the $O(d,d;\mathrm{R})$ and $O(d,d+16;\mathrm{R})$ deformations on the background fields.

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1 Introduction

String theories with extended supersymmetry on the worldsheet are important because of the restrictions which the extra global supersymmetries impose on the target space geometry [1, 2, 3, 4, 5] as well as their natural relation to space-time supersymmetry [6]. The study of N=2 theories, and the profound structures associated with them has been an area of intense investigation in recent years. On the other hand, it is well known that the string vacua, considered as conformal field theories, in general, have deformations under which the geometry of the target space changes. An important case is when the background fields have translational invariance along d of the space coordinates. The string vacua, then, admit deformations generated by the group $O(d, d; \mathrm{R})$ [9, 10, 11, 12], which, in heterotic string theory with abelian gauge fields is generalized to $O(d, d + 16; \mathrm{R})$ [13][14]. This is a generalization of Narain's group to space-time dependent background fields [7][8]. These deformations include the discrete duality transformations which are symmetries of the underlying conformal field theory [15][16][11]. For an exhaustive set of references on this subject see [17]. Now, consider a non-linear σ-model with extended supersymmetry on the worldsheet which also admits $O(d, d)$ (or $O(d, d + 16)$) deformations. These deformations change the background fields in a highly non-trivial manner and, therefore, interfere with the constraints which the extended worldsheet supersymmetry imposes on the target space geometry. The issue of interest is to study the effect of the deformations of the background fields on the extended worldsheet supersymmetry.

The $N = 1$ local supersymmetry on the worldsheet does not impose any restrictions on the background fields and is, therefore, trivially preserved under the above mentioned deformations. The extended supersymmetries, however, require the existence of complex structures on the target manifold thereby restricting its geometry. If the theory also has some isometries, then it admits deformations which change the target space geometry and field configuration, while, preserving the worldsheet conformal invariance. It has been known for sometime that theories with extended $N = 2$ supersymmetry which can be represented in terms of chiral and twisted chiral superfields and which also have an isometry, admit a discrete duality transformation [2]. This duality transformation converts a chiral superfield into a twisted chiral one (or *vice versa*) and also changes the target space geometry. It has been shown that this duality is the same as the usual duality transformation which is a discrete subgroup of the $O(d, d)$ group when the latter is applied to $N = 2$ theories which admit a manifest superfield representation[16][18]. A generic class of such theories in four dimensions were considered in [18] and the duality transformation was combined with gauge transformations of the anti-symmetric tensor field to produce a family of non-trivial $O(d, d)$ deforamations of the background fields. Since at any step the transformation is compatible with supersymmetry, it is insured that the one-parameter family of theories thus obtained also has an extended supersymmetry. However, these theories may no longer have a representation in terms of chiral and twisted chiral superfields. In [19], duality with respect to one isometry direction was considered in the more general case, when a superfield representation

is not necessarily known, and the dual complex structures were obtained. Strictly speaking, these results are valid only when the complex structures do not depend on the coordinate with respect to which duality is performed [1]. Since non-trivial $O(d,d)$ transformations can be obtained by intertwining duality transformations with general coordinate transformations and gauge transformations of the anti-symmetric tensor field, it is expected that they should also preserve the extended worldsheet supersymmetry. However, the action of a generic, non-trivial $O(d,d)$ or $O(d,d+16)$ transformation on the complex structures and the associated supersymmetry is not clear. In this paper, we undertake a study of this problem by investigating the effect of deformations (including duality transformations) of the superstring and heterotic string backgrounds on the constraints that the supersymmetries impose on the target manifold. However, we restrict ourselves to the case when the complex structures are independent of the d coordinates with respect to which the deformations are performed. We obtain the transformation properties of the complex structures and show that the deformations preserve extended supersymmetries on the worldsheet. Hence, they correspond to marginal deformations of the underlying superconformal field theory. They, however, do not preserve the Kähler structure [1] and the product structure [2][4] on the target manifold. The explicit form of the transformations we have used are accurate to one-loop level in the σ-model perturbation theory, although corrections are known to exist to all orders [10][13]. The calculation is simplified by introducing some quantities which transform nicely under $O(d,d)$ and $O(d,d+16)$ deformations.

The paper is organized as follows: In section 2, we linearize the action of the $O(d,d)$ transformations using the target space vielbeins. We, then, construct two connections induced by these transformations which are related by the worldsheet parity transformation. In section 3, we consider non-linear σ models with extended $N=2$ and $N=4$ supersymmetries and translational invariance along d of the target space coordinates. We find the transformations of the complex structures under $O(d,d)$ deformations and using the connections introduced in section 2, show that the deformations preserve the extended worldsheet supersymmetry. In section 4, we consider, in more detail, the transformations of the complex structures in relation to manifest superfield representation of theories with extended $N=2$ supersymmetry. It is shown that, generically, the deformed theories do not have a manifest superfield representation. The case of discrete duality transformation with respect to d isometries is discussed with reference to its relation to a duality known in the $N=2$ theories. In section 5, we linearize the action of the $O(d,d+16)$ group on the heterotic string backgrounds and, using the induced connection, show the invariance of the extended $(0,2)$ and $(0,4)$ supersymmetries under the deformations. The results are summarized in section 6.

[1]See the note added at the end regarding the recent results when this is not the case.

2 Linearization of the $O(d,d)$ Transformations and the Induced Connections

In this section we will rewrite the action of the $O(d,d)$ group, which deforms string vacua with d isometries, in terms of the target space vielbeins and review its relevant aspects for later reference. Then, we introduce two matrices, $Q_{\pm}$, which implement the deformations on the background fields and construct two $O(d,d)$ induced connections. These will be used in our analysis of extended worldsheet supersymmetry in section 3.

It is known that when the background fields, $G_{MN}(X)$, $B_{MN}(X)$ and $\Phi(X)$ are independent of some d (out of D) of the space-time coordinates, then the low-energy effective action is invariant under a set of non-trivial transformations which are generated by the non-linear action of an $O(d,d)$ group on the background fields [9][10][11]. In the case of heterotic string theory, this deformation group is enlarged to $O(d,d+p)$ for a configuration of the gauge fields which commutes with p of the Cartan generators of the gauge group [13][14]. These transformations, on one hand, change the geometry of space-time and, on the other, generate deformations in the underlying conformal field theory [12]. The corresponding conformal field theory moduli can be identified with quantities like the axion charge, angular momentum and the electric charge in the associated space-time theory. In this section we will discuss the $O(d,d)$ transformations in superstring theory and will return to the case of heterotic string theory in section 5. The conventions adopted are as follows: The D-dimensional space-time indices are written as $K, L, M, \ldots$ and the corresponding tangent space indices are written as $A, B, C, \ldots$. All fields are assumed to be independent of d of the space coordinates with indices denoted by $l, m, n, \ldots$. The remaining $D-d$ coordinates, on which the background fields can depend, include the time direction and carry indices $\mu, \nu, \gamma, \ldots$. We will, however, use the matrix notation and suppress the indices whenever possible. In that case, the first d rows and d columns are labeled by indices $l, m, n, \ldots$ and the remaining ones are labeled by the indices $\mu, \nu, \gamma, \ldots$.

The object which transforms nicely under the $O(d,d)$ transformations is a $2D$ dimensional matrix, M, constructed out of the background fields G_{MN} and B_{MN} as

$$M = \begin{pmatrix} G^{-1} & 1_D - G^{-1}B \\ 1_D + BG^{-1} & G - BG^{-1}B \end{pmatrix} \tag{1}$$

To obtain an action on M, the group $O(d,d)$ is considered as a subgroup of the larger group $O(D,D)$ in its fundamental representation. If Ω is an element of the $O(d,d)$ group constructed in this way, then the matrix M transforms under the adjoint action of Ω,

$$\widetilde{M} = \Omega M \Omega^T, \qquad \Omega \in O(d,d) \subset O(D,D) \tag{2}$$

Since the fields G_{MN} and B_{MN} are uniquely determined by M, their transformation can be obtained from that of M. At the one-loop level in the σ-model perturbation theory, this

is also supplemented by an appropriate transformation of the dilaton field Φ which is not relevant for our purposes. The representation is chosen such that the equation defining the group elements Ω takes the form

$$\Omega^T L\,\Omega = L\,, \quad L = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix} \tag{3}$$

This shows that for the purpose of extracting the transformations of the background fields, M is arbitrary up to the addition of a multiple of L.

The embedding of Ω in $O(D,D)$ is given by the following parametrization

$$\Omega = \begin{pmatrix} A_1 & C_1 \\ C_2 & A_2 \end{pmatrix};\ A_i = \begin{pmatrix} \mathcal{A}_i & 0 \\ 0 & 1_{D-d} \end{pmatrix},\ C_i = \begin{pmatrix} \mathcal{C}_i & 0 \\ 0 & 0_{D-d} \end{pmatrix}, \tag{4}$$

where,

$$\begin{pmatrix} \mathcal{A}_1 & \mathcal{C}_1 \\ \mathcal{C}_2 & \mathcal{A}_2 \end{pmatrix} \in O(d,d)$$

A general $O(d,d)$ transformation can be parametrized in terms of its action on the background fields. The group elements given by $\mathcal{A}_2 = (\mathcal{A}_1^T)^{-1}, \mathcal{C}_1 = \mathcal{C}_2 = 0$, correspond to $GL(d,R)$ transformations and the ones given by $\mathcal{A}_2 = \mathcal{A}_1 = 1, \mathcal{C}_1 = 0, \mathcal{C}_2 = -\mathcal{C}_2^T$, correspond to constant gauge transformations of the antisymmetric tensor field, B_{MN}. The non-trivial deformations of the backgrounds are generated by elements from $O(d) \times O(d)/O(d)$. The $O(d) \times O(d)$ subgroup is parametrized as

$$\Omega = \frac{1}{2} \begin{pmatrix} S+R & S-R \\ S-R & S+R \end{pmatrix}, \text{ where, } S(R) = \begin{pmatrix} \mathcal{S}(\mathcal{R}) & 0 \\ 0 & 1_{D-d} \end{pmatrix},\ \mathcal{S}, \mathcal{R} \in O(d). \tag{5}$$

The diagonal $O(d)$ subgroup, given by $\mathcal{S} = \mathcal{R}$, generates ordinary rotations and is already included in $GL(d,R)$. To get non-trivial deformations, one considers (5) modulo this subgroup. The usual duality transformations [15] with respect to isometries along coordinates X^i, are contained in a discrete subgroup generated by

$$\mathcal{S} = 1\,,\ \mathcal{R} = 1 - 2\varepsilon_i\,, \quad \text{where, } (\varepsilon_i)_{jk} = \delta_{ij}\delta_{ik} \tag{6}$$

The full generalized duality group, $O(d,d;\mathrm{Z})$, is generated by (6), along with $GL(d,\mathrm{Z})$ and discrete gauge transformations of B_{MN} [11]. In the following, however, we will only refer to (6) as duality transformations.

To simplify the manipulations, in the following, we rewrite the above transformations in terms of the metric vielbein. Using the inverse vielbein, e, and a matrix K defined as

$$G^{-1} = e\,\eta\,e^T, \qquad K = G + B, \tag{7}$$

we construct a $2D \times D$ rectangular matrix ξ as

$$\xi = \begin{pmatrix} e \\ Ke \end{pmatrix} \tag{8}$$

This is related to M by

$$M = \xi\, \eta\, \xi^T = \begin{pmatrix} e \\ Ke \end{pmatrix} \eta \begin{pmatrix} e^T & e^T K^T \end{pmatrix}$$

and transforms, under $O(d,d)$, as

$$\widetilde{\xi} = \Omega\, \xi\,, \qquad \Omega \in O(d,d) \subset O(D,D) \tag{9}$$

Note that the Lorentz indices are not affected by the transformations[2]. In the remaining part of this section, we will describe some constructions which are used for our analysis of extended worldsheet supersymmetry in section 3.

Since the extended worldsheet supersymmetries are manifestly invariant under $GL(d,R)$ and the B_{MN} gauge transformations, in the following, we will mainly concentrate on the non-trivial deformations generated by the $O(d) \times O(d)$ subgroup. As argued in the previous section, it is expected that these deformations also preserve the extended supersymmetry, though their effect on the complex structures and, therefore, on the supersymmetry charges is far from clear. Under the action of this subgroup, which is parametrized by the matrices S and R, the vielbein e, the metric G, and the matrix $K = G + B$ transform as

$$\begin{array}{rcl} \widetilde{e} &=& Q_-(S,R)\, e \\ \widetilde{G}^{-1} &=& Q_-(S,R)\, G^{-1} Q_-^T(S,R) \\ \widetilde{K} &=& Q_-(S,-R)\, Q_-^{-1}(S,R) \end{array} \tag{10}$$

where,

$$\begin{aligned} Q_-(S,R) &= \frac{1}{2}\Big(S + R + (S-R)\, K \Big) \\ Q_-^{-1}(S,R) &= \frac{1}{2}\Big(S^T + R^T + (S^T - R^T)\, \widetilde{K} \Big) \end{aligned} \tag{11}$$

The second equation above follows from the fact that $\Omega^{-1}(S,R) = \Omega(S^T, R^T)$ and the subscript for Q has been chosen in anticipation of the transformation properties of the complex structures to be discussed in the next section. Note that though $O(d,d)$ is a global transformation, the matrix Q_- which implements the transformation becomes local through its dependence on $K(X) = G(X) + B(X)$. Also, since the conditions for the existence of extended supersymmetry on the worldsheet involve space-time derivatives of the background

[2] When $S = R$, one may perform compensatory transformations on the Lorentz indices so that the flat vielbein is invariant under rotations.

fields, it is desirable to find convenient expressions for the transformation of such quantities. To this end, first note that under the transformation (9), the quantity $\xi^T L\,\partial_\mu\xi$ is invariant. Now, consider a vector field $V(X)$ which under $O(d,d)$ transforms as $\widetilde{V} = Q_- V$. Corresponding to this transformation, we can construct an $O(d,d)$ induced connection [3] as $e\,\partial_\mu e^{-1}$. However, it is more useful to add an $O(d,d)$covariant piece to this expression and construct a new induced connection, ω^-_μ, given by

$$\omega^-_\mu = \frac{1}{2}\,e\eta\xi^T L\partial_\mu\xi e^{-1} - e\partial_\mu e^{-1}, \tag{12}$$

in terms of which,

$$\partial_\mu K = 2\,G\,\omega^-_\mu \tag{13}$$

Under an $O(d,d)$ deformation, this connection transforms to

$$\widetilde{\omega}^-_\mu = Q_-\,\omega^-_\mu Q_-^{-1} - \partial_\mu Q_- Q_-^{-1} \tag{14}$$

It should be emphasized that ω^-_μ is a connection in the space-time sense but is not the same as the natural torsion-full connections on a σ-model manifold (see equation (29) below).

On the worldsheet, the extended supersymmetries in the left moving sector and the right moving sector are interchanged by the worldsheet parity transformation, $\sigma \to -\sigma$. Under this transformation, $B \to -B$ and S and R are interchanged. The matrix Q_-, therefore, goes over to a new matrix Q_+, given by

$$\begin{aligned} Q_+ &= \frac{1}{2}\Big(\,S + R - (S-R)\,K^T\,\Big) \\ Q_+^{-1} &= \frac{1}{2}\Big(\,S^T + R^T - (S^T - R^T)\,\widetilde{K}^T\Big) \end{aligned} \tag{15}$$

Since G is invariant under the worldsheet parity, one expects $\widetilde{G}$ also to be invariant. In fact, using (11),(15) and (5), it can be shown that

$$\widetilde{G}^{-1} = Q_-\,G^{-1}\,Q_-^T = Q_+\,G^{-1}\,Q_+^T \tag{16}$$

Now, we introduce a second connection corresponding to transformations of the form $\widetilde{V} = Q_+ V$. This connection, the relevance of which will become clear in the next section, is given by

$$\omega^+_\mu = G^{-1}\,\omega^{-T}_{\ \mu}\,G, \tag{17}$$

and, under an $O(d,d)$ deformation, transforms as

$$\widetilde{\omega}^+_\mu = Q_+\,\omega^+_\mu Q_+^{-1} - \partial_\mu Q_+ Q_+^{-1} \tag{18}$$

[3] As expected, this connection has zero curvature.

Having introduced two connections, we can also define the corresponding $O(d,d)$ covariant derivatives,

$$\mathcal{D}^{\pm}_{\mu} = \partial_{\mu} + \omega^{\pm}_{\mu}. \tag{19}$$

The actual form of the covariant derivative, obviously, depends on the particular representation chosen. At the end, note that the transformations of the two connections, (14) and (18), can also be written as

$$\tilde{\omega}^{\pm}_{\mu} = Q_{\mp}\omega^{\pm}_{\mu}Q^{-1}_{\pm}. \tag{20}$$

3 Extended Supersymmetry on the Worldsheet and the $O(d,d)$ Deformations

In this section, we obtain the transformations of the complex structures under $O(d,d)$ and show that these deformations of the background fields preserve the conditions of extended $(2,2)$ and $(4,4)$ supersymmetries on the worldsheet. We will first consider the $(2,2)$ and then the $(4,4)$ extended supersymmetries. The result for the left-right asymmetric situations follows trivially. We will also comment on the possible restrictions on the form of perturbative corrections to the $O(d,d)$ transformation equations in the context of $(4,4)$ supersymmetry.

A non-linear σ-model with local $N=1$ supersymmetry, in both the left and right moving sectors, can have a second supersymmetry in both sectors provided the target manifold admits two complex structures J^+ and J^- [1][2][6][4][5]. The second supersymmetry transformations are obtained from the first one by replacing the worldsheet fermions, ${\psi_{\pm}}^M$, by ${J^{\pm}}^M{}_N{\psi_{\pm}}^N$ in the supersymmetry transformation equations. The linear independence of the two supersymmetries and the invariance of the σ-model action imposes the following restrictions on the target manifold:

$$ {J^{\pm}}^M{}_N {J^{\pm}}^N{}_K = -\delta^M_K \tag{21}$$

$$ {N^{\pm}}^K{}_{MN} = {J^{\pm}}^L{}_M \partial_{[L}{J^{\pm}}^K{}_{N]} - {J^{\pm}}^L{}_N \partial_{[L}{J^{\pm}}^K{}_{M]} = 0 \tag{22}$$

$$ {J^{\pm}}^M{}_K G_{MN} {J^{\pm}}^N{}_L = G_{KL} \tag{23}$$

$$ \nabla^{\pm}_M {J^{\pm}}^N{}_K = \partial_M {J^{\pm}}^N{}_K + {\Omega^{\pm}}^N{}_{ML}{J^{\pm}}^L{}_K - {\Omega^{\pm}}^L{}_{MK}{J^{\pm}}^N{}_L = 0 \tag{24}$$

Where, $\Omega^{\pm}$ are non-Riemannian connections constructed in terms of the Christoffel symbol and the torsion tensor,

$$\begin{aligned} {\Omega^{\pm}}^N{}_{ML} &= \mathit{\Gamma}^N_{ML} \pm G^{NP}\, H_{PML} \\ &= \frac{1}{2}G^{NP}\left[\partial_M(G\mp B)_{PL} + \partial_L(G\mp B)_{MP} - \partial_P(G\mp B)_{ML}\right] \end{aligned} \tag{25}$$

Equations (21) - (23) mean that the manifold admits two integrable almost complex structures both of which give rise to hermitian structures. Equation (24) is the condition for covariant constancy of the complex structures with respect to the generalized connections given in (25). These are the most general conditions for the existence of (2,2) extended supersymmetry on the worldsheet [5]. Note that in the absence of torsion, the manifold is Kähler and the two complex structures can be chosen to be the same. These theories have a manifest supersymmetric representation in terms of chiral superfields [1]. A more general situation arises when the torsion is not zero, but the complex structures, J^+ and J^-, commute with each other. These theories can be written in manifestly supersymmetric form in terms of the chiral and twisted chiral superfields [2][4]. When the two complex structures do not commute, a superfield representation is not known in general.

To investigate the invariance of the above conditions under $O(d,d)$ deformations, we need the transformation properties of the complex structures $J^{\pm}$ under the deformations. These can be obtained by using the transformation of the metric as given in (16) and demanding that the deformations preserve the hermitian structures (23). At this level, however, there is an ambiguity in determining the transformations of the complex structures because of the two possible ways in which a transformation of the metric can be written in (16). One way of resolving the ambiguity, obviously, is to find out which form of the transformations, if any, keeps the conditions (24) invariant. But, a simpler and more illuminating method is to note that for infinitesimal background fields, $G = 1 + h, B = b$, equation (10) reduces to $\widetilde{h} + \widetilde{b} = R(h + b)S^T$. The quantity $h + b$ can also be interpreted as the polarization tensor in a vertex operator for the emission of massless states in superstring theory formulated in flat backgrounds [10][13]. If we consider correlation functions of this vertex operator which carry zero momentum along d of the space coordinates, then, the above transformation is equivalent to rotating, by different amounts, the left-moving and right-moving parts of the d bosonic coordinates along with the left-moving and right moving parts of their fermionic partners: $(\,\partial X, \psi_+\,) \rightarrow (\,R\partial X, R\psi_+\,)$ and $(\,\bar{\partial} X, \psi_-\,) \rightarrow (\,S\bar{\partial} X, S\psi_-\,)$. In order to keep the second supersymmetry intact, this implies that to zeroth order in the backgrounds, the complex structures should transform as $\widetilde{J}^+ = RJ^+R^T$ and $\widetilde{J}^- = SJ^-S^T$. Also in flat backgrounds, $Q_- = S$ and $Q_+ = R$. Combining this fact with (16), and requiring the invariance of the hermiticity condition (23), the transformations of the complex structures under $O(d,d)$ are uniquely determined as

$$\begin{aligned} \widetilde{J}^- &= Q_- J^- Q_-^{-1} \\ \widetilde{J}^+ &= Q_+ J^+ Q_+^{-1} \end{aligned} \tag{26}$$

These transformations trivially preserve the almost complex structures on the manifold (21). In the following, we analyze the covariance of conditions (22) and (24). We restrict ourselves to $J^{\pm}$ that are independent of the d coordinates X^m. For the case when this is not true, see the note added at the end.

To simplify the analysis, we introduce two rank 3 $O(d,d)$ tensors, $\mathcal{J}^{\pm\lambda}$, defined as

$$\mathcal{J}^{\pm\lambda LN}_{\;\;K} = G^{LN} J^{\pm\lambda}{}_K - J^{\pm L}{}_K G^{N\lambda} - \delta^\lambda{}_K (J^\pm G^{-1})^{NL} + \delta^L{}_K (J^\pm G^{-1})^{N\lambda}, \tag{27}$$

Under an $O(d,d)$, they transform to

$$\widetilde{\mathcal{J}}^{\pm\lambda L'N'}_{\quad K'} = Q_{\pm}{}^{L'}_{\ L}\, Q_{\pm}{}^{N'}_{\ N}\, Q_{\pm}^{-1}{}^{K}_{\ K'}\, \mathcal{J}^{\pm\lambda LN}_{\quad K}, \tag{28}$$

as can be seen using (10) and (26). Also note that in terms of the induced connections $\omega^{\pm}_{\mu}$ given in (12) and (17), the generalized connections (25) take the form

$$\Omega^{\pm N}_{\ \ ML} = \omega^{\pm}_{M}{}^{N}_{\ L} + (G\,\omega^{\pm}_{L}G^{-1})_{M}^{\ N} - G^{N\mu}(G\,\omega^{\pm}_{\mu})_{ML} \tag{29}$$

Using the above relations and the $O(d,d)$ tensors (27), the conditions of covariant constancy of the complex structures (24) can be rewritten as the following two equations (for $M = m, \mu$),

$$\begin{aligned} M = m: &\qquad (\nabla^{\pm}_{m}J^{\pm})^{N}_{\ K} = (G\,\omega^{\pm}_{\lambda})_{mL}\, \mathcal{J}^{\pm\lambda LN}_{\quad K} = 0 \\ M = \mu: &\qquad (\nabla^{\pm}_{\mu}J^{\pm})^{N}_{\ K} = (\mathcal{D}^{\pm}_{\mu}J^{\pm})^{N}_{\ K} + (G\,\omega^{\pm}_{\lambda})_{\mu L}\, \mathcal{J}^{\pm\lambda LN}_{\quad K} = 0. \end{aligned} \tag{30}$$

Here,

$$\mathcal{D}^{\pm}_{\mu}J^{\pm} = \partial_{\mu}J^{\pm} + [\,\omega^{\pm}_{\mu}\,,\, J^{\pm}\,]$$

are the two $O(d,d)$ induced covariant derivatives introduced in the previous section (19). The analysis of the the invariance of these conditions under an $O(d,d)$ deformation is now straightforward. Using the transformation laws of the connections $\omega^{\pm}_{\mu}$ (14)(18) and of tensors $\mathcal{J}^{\pm\lambda}$ (28), along with the fact that $(Q_{\pm})^{\mu}_{\ m} = (Q^{-1}_{\pm})^{\mu}_{\ m} = 0$ and $(Q_{\pm})^{\mu}_{\ \nu} = \delta^{\mu}_{\nu}$, we obtain

$$\begin{aligned} M' = m': &\qquad (\widetilde{\nabla}^{\pm}_{m'}\widetilde{J}^{\pm})^{N'}_{\ K'} = Q_{\mp}^{-1}{}^{m}_{\ m'}\, Q_{\pm}{}^{N'}_{\ N}\, Q_{\pm}^{-1}{}^{K}_{\ K'} (\nabla^{\pm}_{m}J^{\pm})^{N}_{\ K} = 0 \\ M' = \mu: &\qquad (\widetilde{\nabla}^{\pm}_{\mu}\widetilde{J}^{\pm})^{N'}_{\ K'} = Q_{\pm}{}^{N'}_{\ N}\, Q_{\pm}^{-1}{}^{K}_{\ K'} \Big[(\nabla^{\pm}_{\mu}J^{\pm})^{N}_{\ K} + Q_{\mp}^{-1}{}^{m}_{\ \mu} (\nabla^{\pm}_{m}J^{\pm})^{N}_{\ K} \Big] = 0 \end{aligned} \tag{31}$$

where, the vanishing of the right-hand sides follows from (30). This proves that the transformed complex structures are still covariantly constant with respect to the deformed generalized connections. The last equations to check are the integrability conditions (22) of the almost complex structures $J^{\pm}$. After some manipulations, the $O(d,d)$ deformed Nijenhuis tensors can be written as

$$\widetilde{N}^{\pm K'}_{\ \ M'N'} = Q_{\pm}^{-1}{}^{M}_{\ M'}\, Q_{\pm}^{-1}{}^{N}_{\ N'}\, Q_{\pm}{}^{K'}_{\ K} \left(N^{\pm K}_{\ \ MN} + \mathcal{J}^{\pm\lambda LP}_{\quad M}\, G_{PN} \left[\, Q^{-1}_{\pm}\partial_{\lambda}Q_{\pm}\,,\, J\,\right]^{K}_{\ L} \right)$$

which, on further manipulation, reduce to

$$\begin{aligned} \widetilde{N}^{\pm K'}_{\ \ M'N'} &= Q_{\pm}^{-1}{}^{M}_{\ M'}\, Q_{\pm}^{-1}{}^{N}_{\ N'}\, Q_{\pm}{}^{K'}_{\ K} \Big(N^{\pm K}_{\ \ MN} \\ &\quad \mp \left(\delta^{K}_{L}(GJ^{\pm})_{PN} - J^{\pm K}_{\ \ L}G_{PN}\right) \left(Q^{-1}_{\pm}\,(S-R)\right)^{Lm} (\nabla^{\pm}_{m}J^{\pm})^{P}_{\ M} \Big) = 0 \end{aligned} \tag{32}$$

Here, the vanishing of the deformed Nijenhuis tensors follows from equations (22) and (24). Thus, the deformed almost complex structures are integrable. This completes the proof

of the invariance of the $(2,2)$ extended supersymmetry on the worldsheet under $O(d,d)$ deformations.

The only other possible extension of the $N=1$ supersymmetry is to $N=4$ [2][4]. In the $(4,4)$ case, this extension requires the existence, in each chiral sector separately, of three complex structures, $J_a; a=1,2,3$. Each one of the J_a's satisfies conditions (21)-(24) independently. In addition, the J_a satisfy an $SU(2)$ algebra, giving rise to a quaternionic structure on the target manifold,

$$J_a^{\pm} J_b^{\pm} = -\delta_{ab} + \epsilon_{abc} J_c^{\pm}.$$

Also, condition (22) is now generalized to the vanishing of the mixed Nijenhuis tensors,

$$N^{\pm\,K}_{(ab)\,MN} = J^{\pm\,L}_{(a\,M}\,\partial_{[L}\,J^{\pm\,K}_{b)\,N]} - J^{\pm\,L}_{(a\,N}\,\partial_{[L}\,J^{\pm K}_{b)\,M]} = 0$$

The invariance of the constraints of extended $(4,4)$ supersymmetry under a deformation follows from the above discussion for the $(2,2)$ case coupled with the fact that the transformations of the complex structures do not affect the $SU(2)$ index a. This proves the invariance of the extended $(4,4)$ supersymmetry under $O(d,d)$ deformations when all complex structures are independent of the d coordinates X^m. Since supersymmetry is preserved in the left and right chiral sectors independently, the above analysis can be trivially generalized to any model with extended (p,q), $p,q=0,1,2,4$ supersymmetry.

The invariance of the $(4,4)$ supersymmetry under the deformations may have an implication for the possible form of higher σ-model loop corrections to the $O(d,d)$ transformations. The form of the transformation given in (2), with M defined as in (1), is correct to one-loop in the σ-model perturbation theory. However, the arguments in [10],[13] and the analysis of [11] and [12] imply the existence of corrections, to all orders, to the transformation. On the other hand, if the non-renormalization theorems for the extended $(4,4)$ supersymmetry on the worldsheet [4] are valid, then, for these theories, the corrections to the $O(d,d)$ transformations must vanish. This restricts the possible form of the higher loop corrections in terms of the constraints which the $(4,4)$ supersymmetry imposes on the background fields.

4 Complex Structures, Duality and Manifest $N=2$ Supersymmetry

In this section we analyse the transformations of the complex structures and show that, starting from a theory with manifest $N=2$ supersymmetry, the deformed theories do not, in general, admit a manifestly supersymmetric description in terms of chiral and twisted chiral superfields. We write the action of a general discrete duality transformation on the complex structures and discuss the case of Kähler manifolds.

The existence of extended $N = 2$ supersymmetry on the worldsheet severely constrains the target space geometry by requiring the existence of two complex structures $J^{\pm}$ satisfying equations (21)-(24). A simpler situation arises when the torsion, H_{MNP}, on the target space is zero. Conditions (21)-(24), then, do not distinguish between the two complex structures which can, therefore, be chosen to be the same. This complex structure is covariantly constant with respect to the Riemmanian connection and the corresponding manifold is Kähler. A non-linear σ-model defined on a Kähler manifold can be written, in a manifestly $(2,2)$ supersymmetric form, in terms of the Kähler potential as a function of $N = 2$ chiral superfields. In the presence of torsion, a superfield formulation of the theory is not always known. However, there is a special class of theories with $H_{MNP} \neq 0$ which still admit a manifestly supersymmetric description in terms of chiral and twisted chiral superfields. These are the theories in which the two complex structures commute and are, therefore, simultaneously diagonalizable [2][4]. We can always choose a canonical basis in which J^- takes the form ${J^-}^a{}_b = i\delta^a{}_b$, ${J^-}^{\bar{a}}{}_{\bar{b}} = -i\delta^{\bar{a}}{}_{\bar{b}}$, where, $a, \bar{a} = 1, \ldots, D/2$ label the holomorphic and antiholomorphic coordinates. In this basis, J^+ is also diagonal but the arrangement of $\pm i$'s on the diagonal depend on the number of twisted chiral superfields or, equivalently, the form of B_{MN}. Such a manifold is said to have a product structure.

The Kähler and product structures, however, are very special and are not, in general, preserved under non-trivial $O(d,d)$ deformations of the complex structures. To investigate this point, we first consider the case of flat backgrounds. In this case, as discussed above equation (26), the transformations can be interpreted as independent d-dimensional rotations of the left and right moving parts of the bosonic and fermionic coordinates inside correlation functions which carry zero momentum along d of the space directions. Modulo ordinary d-dimensional rotations and parity transformations, which are symmetries of the theory, we can choose $S = 1$ in (26). The complex structures then transform as $J^- \to J^-$, $J^+ \to RJ^+R^T$. Note that R is rotation a involving the real coordiantes. Therefore, if we choose the usual real representation for R, then, J^+ also has to be written in the real coordinate basis. If we write J^+ in the diagonal basis (which is not real), then R also has to be transformed appropriately. It can be explicitly checked that starting from commuting complex structures, non-trivial R transformations that preserve this commuting nature are (i) the ones which flip the $+i$ and $-i$ eigenvalues of J^+ written in the canonical basis of J^-, (ii) the ones that correspond to $O(2)$ rotations involving the real and imaginary parts of the same complex coordinate (note that a rotation which mixes the real and imaginary parts of different complex coordinates does not qualify). The first case corresponds to converting a chiral superfield into a twisted chiral one, or *vice versa* and it can be checked that these transformations form the discrete duality subgroup (6) of $O(d,d,R)$. This once again shows the connection between the usual duality and the $N = 2$ duality of [2]. Case (ii) corresponds to $O(2) \times O(2)/O(2)$ deformations. The remaining elements of $O(d) \times O(d)/O(d)$ do not, in general, preserve the commuting nature of the complex structures even in flat background fields. Therefore, in the following, where the background fields are not flat, we will concentrate only on cases (i) and (ii) above.

In the presence of non-trivial background fields, it can be explicitly checked that $O(2) \times$

$O(2)/O(2)$ deformations do not generically preserve the commuting nature of the complex structures. One exception is when the manifold is Kähler and the metric is independent of one of the complex coordinates, say z_1, and its conjugate. then it can be explicitly checked that an $O(2,2)$ deformation involving only the real and imaginary parts of z_1 does not change the complex structures. The manifold, thus, remains Kähler even after the deformation. We, now, turn to the case of discrete duality transformations (6). A duality with respect to d of the coordinates, on which the background fields do not depend, is generated by $\mathcal{S} = 1\,, \mathcal{R} = 1 - 2\,\sum_{i=1}^{d} \varepsilon_i$ [15][16][11][8]. The dual complex structures, obtained from (26), are given by

$$\widetilde{J}^- = \begin{pmatrix} (KJ^-)^m_{\ l}(\mathcal{K}^{-1})^l_{\ n} & -(KJ^-)^m_{\ l}(\mathcal{K}^{-1})^l_{\ p}K^p_{\ \nu} + (KJ^-)^m_{\ \nu} \\ J^{-\mu}_{\ \ l}(\mathcal{K}^{-1})^l_{\ n} & -J^{-\mu}_{\ \ l}(\mathcal{K}^{-1})^l_{\ p}K^p_{\ \nu} + J^{-\mu}_{\ \ \nu} \end{pmatrix} \tag{33}$$

and

$$\widetilde{J}^+ = \begin{pmatrix} (K^TJ^+)^m_{\ l}(\mathcal{K}^{-1T})^l_{\ n} & (K^TJ^+)^m_{\ l}(\mathcal{K}^{-1T})^l_{\ p}K^{Tp}_{\ \ \nu} - (K^TJ^+)^m_{\ \nu} \\ -J^{+\mu}_{\ \ l}(\mathcal{K}^{-1T})^l_{\ n} & -J^{+\mu}_{\ \ l}(\mathcal{K}^{-1T})^l_{\ p}K^{Tp}_{\ \ \nu} + J^{+\mu}_{\ \ \nu} \end{pmatrix} \tag{34}$$

where, $\mathcal{K}_{mn} = K_{mn}\,; m, n = 1, \ldots, d$ is a $d \times d$ submatrix of K, and the indices of K are raised and lowered by a flat metric which appears in the $O(d,d)$ transformations and has not been explicitly written here. For $d = 1$, these expressions were obtained in [19]. From the above, one can see that even if the complex structures, J^+ and J^-, commute, their duals in general do not commute except for some restricted backgrounds. A situation which can be discussed in some generality is the Kähler manifold with isometries, on which, the two complex structures are chosen to be equal. If the isometries are along some of the real coordinates which define the canonical basis for the complex structure, then, one can explicitly check that the dual complex structures are still commuting. This reproduces the duality of $N = 2$ theories discovered in [2] when the starting theory is defined on a Kähler manifold. However, for more general isometries, after a duality transformation with respect to d isometries, the dual complex structures do not commute, even for this restricted class. The non-zero terms in the commutator are, however, proportional to the components of the Kähler form, GJ, in the isometry directions. Since the Kähler form is antisymmetric, for $d = 1$ the dual complex structures commute without further restrictions on the background fields.

From the above discussion it follows that though the deformations preserve the extended $N = 2$ supersymmetry, the Kähler and product structures are not, in general, preserved. In such cases, the deformed theories do not have a formulation in terms of chiral and twisted chiral superfields. This implies that, starting from a conformally invariant non-linear σ-model with manifest extended $N = 2$ supersymmetry and some isometries, one can construct a large class of new theories with $N = 2$ extended supersymmetry, for which, a superfield representation may not be known. This is achieved without solving constraints (21)-(24) and the β-function equations. Duality transformations applied to the models of [2] preserve the

superfield representation. The discussion also clarifies the connection between the duality transformations (6) and the duality of $N=2$ theories discussed in [2].

5 The $O(d,d+16)$ Deformations and Extended World-sheet Supersymmetry in Heterotic String Theory

In this section we rewrite the action of the $O(d,d+16)$ group on the heterotic string theory backgrounds, including the gauge Chern-Simons term, in terms of the metric vielbein. We then construct a connection induced by these transformations and show that the extended worldsheet supersymmetry in heterotic string theory is preserved under the deformations.

In heterotic string theory, the $O(d,d)$ group of deformations of the string vacua with d isometries is enlarged to an $O(d,d+p)$ group provided all the background gauge fields belong to a subgroup that commutes with p of the Cartan generators of the gauge group [7][13][14]. For simplicity, we consider the case when $p=16$ and, therefore, all background gauge fields are abelian. As in the non-heterotic case, the action of the $O(d,d+16)$ group on the background fields is given in terms of a $(2D+16)\times(2D+16)$ dimensional matrix M constructed from G_{MN}, B_{MN} and the gauge fields A^I_M, where, $I=1,\ldots,16$ is the gauge group index [13]. The matrix M transforms under the adjoint action of the $O(d,d+16)$ group embedded in the fundamental representation of $O(D,D+16)$. The transformations of the background fields are uniquely determined from that of M. At the one-loop level, this is also accompanied by a transformation of the dilaton field Φ. As in section 2, we rewrite the transformations, in a slightly modified form, in terms a $(2D+16)\times(D)$-dimensional matrix ξ defined as

$$\xi = \begin{pmatrix} e \\ K\,e \\ -A\,e \end{pmatrix} \tag{35}$$

where, A is a $16\times D$-dimensional matrix and K is given by

$$K_{MN} = G_{MN} + B_{MN} + \frac{1}{2}\,A^I_M\,A^I_N \tag{36}$$

The matrix M, mentioned above, can now be constructed in terms of ξ as[4]

$$M = \xi\,\eta\,\xi^T = \begin{pmatrix} e \\ Ke \\ -Ae \end{pmatrix} \eta \begin{pmatrix} e^T & e^T K^T & -e^T A^T \end{pmatrix} \tag{37}$$

Under an $O(d,d+16)$ transformation, ξ transforms as

$$\widetilde{\xi} = \Omega\xi\,, \quad \Omega \in O(d,d+16) \subset O(D,D+16) \tag{38}$$

[4]The fields here are related to those of [13] by $B\to -B, K\to -K, A\to -A/\sqrt{2}$.

where, the representation is chosen such that the defining equation for Ω takes the form

$$\Omega^T L\, \Omega = L\,, \quad L = \begin{pmatrix} 0 & 1_D & 0 \\ 1_D & 0 & 0 \\ 0 & 0 & -1_{16} \end{pmatrix} \tag{39}$$

These transformations include the $GL(d,R)$ transformations and gauge transformations of B_{MN} and A^I_M. The non-trivial deformations are generated by the elements from $(O(d+16)\times O(d))/O(d)$. These are parametrized as

$$\Omega = \frac{1}{2}\begin{pmatrix} S+R & S-R & -R_1^T \\ S-R & S+R & R_1^T \\ -R_2 & R_2 & 2R_3 \end{pmatrix} \tag{40}$$

modulo the subgroup generated by $S=R$. Here, S and R are again given by equation (5) although it is no longer necessary to have $R\in O(d)$; R_3 is a 16×16-dimensional matrix and $R_{1,2}$ are $16\times D$-dimensional matrices of the form

$$R_{1,2} = \begin{pmatrix} \sqrt{2}\,\mathcal{R}_{1,2} & 0_{16\times(D-d)} \end{pmatrix}\,, \quad \text{where,} \begin{pmatrix} \mathcal{R} & \mathcal{R}_1^T \\ \mathcal{R}_2 & R_3 \end{pmatrix} \in O(d+16)$$

Under the deformations generated by the above elements, the background fields transform as

$$\begin{aligned} \widetilde{e} &= Q\, e \\ \widetilde{K} &= \frac{1}{2}\Big((S-R) + (S+R)\,K - R_1^T\, A\Big)\, Q^{-1} \\ \widetilde{A} &= \frac{1}{2}\Big(R_2 - R_2\, K + 2\, R_3\, A\Big)\, Q^{-1} \end{aligned} \tag{41}$$

where, Q is given by

$$Q = \frac{1}{2}\Big((S+R) + (S-R)\,K + R_1^T\, A\Big) \tag{42}$$

The transformation of the metric can be obtained from that of the inverse vielbein e.

Now, as in section 2, we introduce an $O(d,d+16)$ induced connection as follows. Note that, under $O(d,d+16)$ deformations (38), the quantity $\xi^T\, L\,\partial_\mu\,\xi$ is invariant and, from (41), the quantity $e\,\partial_\mu\, e^{-1}$ transforms as a connection corresponding to transformations of the type $\widetilde{V} = QV$. Using these two quantities, we construct a non-minimal $O(d,d+16)$ induced connection ω_μ as

$$\omega_\mu = \frac{1}{2}\, e\eta\xi^T L\partial_\mu\xi e^{-1} - e\partial_\mu e^{-1}, \tag{43}$$

which, in a more recognizable form, can be written as

$$\omega_\mu = \frac{1}{2}\, G^{-1}\,(\partial_\mu\, K - A^T\,\partial_\mu\, A) \tag{44}$$

Under an $O(d, d+16)$ deformation, ω_μ transforms to

$$\widetilde{\omega}_\mu = Q\,\omega_\mu Q^{-1} - \partial_\mu Q\,Q^{-1} \tag{45}$$

Using the above construction, we set out to prove the invariance of the extended worldsheet supersymmetry in heterotic string theory under the $O(d, d+16)$ deformations.

In heterotic string theory, the local $(0,1)$ worldsheet supersymmetry can be extended to a $(0,2)$ supersymmetry in a way similar to the superstring case [3][6][4]. The target manifold is required to admit an almost complex structure J with vanishing Nijenhuis tensor, $N^K_{MN} = 0$, and a hermitian metric, $J^T\,G\,J = G$. The difference with the superstring case, however, is that the torsion tensor H_{MNK} now also contains a gauge Chern-Simons term[5] coming from the one-loop chiral anomaly, $H_{MNK} = (1/2)(\partial_M B_{NK} + \cdots) + (1/4)(A^I_M F^I_{NK} + \cdots)$, where, the dots denote cyclic permutations and we have restricted ourselves to abelian gauge fields. This modifies the generalized connection and, thus, the condition of covariant constancy of the complex structure. With this modification in mind, the conditions for the existence of extended supersymmetry in heterotic string theory are again given by equations (21)-(24) for $J = J^-$. The only extra condition is the constraint on the gauge field background,

$$F^I_{KL}\,J^L{}_M - F^I_{ML}\,J^L{}_K = 0 \tag{46}$$

The transformation of the complex structure J under an $O(d, d+16)$ deformation is obtained by requiring the covariance of the hermiticity condition of the metric and is given by $\widetilde{J} = Q\,J\,Q^{-1}$, with Q as given by (42). This also preserves $J^2 = -1$. To investigate the covariance of the remaining conditions, it is convenient to write them in terms of the induced connection ω_μ (44) and the rank 3 tensor, $\mathcal{J}^{\lambda LN}_K = \mathcal{J}^{-\lambda LN}_{\;\;K}$ (27). First, consider the covariant constancy condition $\nabla_M J = 0$. It turns out that, even in the presence of the gauge Chern-Simons term, the relation between the generalized connection Ω^- and the $O(d, d+16)$ induced connection (44) is given by equation (29). Therefore, the covariant constancy condition again takes the form given in (30). Next, we consider the condition on the gauge fields (46). It is easily seen that this equation can be rewritten as

$$\partial_\lambda\,A^I{}_L\,\mathcal{J}^{\lambda LN}_K\,G_{NM} = 0 \tag{47}$$

Using the above equations in terms of the $O(d, d+16)$ covariant variables, a straightforward calculation shows that, after a deformation, the tensors $N^K_{MN}, \nabla_M J$ and the left hand side of equation (47) transform into linear combinations of each other and, therefore, remain equal to zero. This proves the invariance of the $(0,2)$ worldsheet extended supersymmetry in heterotic string theory under the $O(d, d+16)$ deformations of the background fields. The generalization to $(0,4)$ supersymmetry follows from the discussion, in section 3, of extended $(4,4)$ supersymmetry in superstring theory.

[5]In general, the torsion also contains the Lorentz Chern-Simons term which is a higher derivative term and will not be considered here.

6 Conclusions

We have shown that the $O(d,d;\mathrm{R})$ deformations of the superstring vacua and the $O(d,d+16;\mathrm{R})$ deformations of the heterotic string vacua can be rewritten, in a simpler way, in terms of the target space vielbeins. Though, these transformations are global in the space-time sense, their non-linear action on the background fields enables us to construct some induced space-time connections. We write down the transformations of the complex structures associated with the extended worldsheet supersymmetries, under the above deformations. The analysis is valid only when the complex structures are independent of the d coordinates with respect which the deformations are performed. Using the induced connections and some tensors which transform covariantly under the deformations, we show that the $O(d,d;\mathrm{R})$ deformations of the superstring vacua and the $O(d,d+16;\mathrm{R})$ deformations of the heterotic string vacua preserve the extended supersymmetries on the worldsheet. They, therefore, generate marginal deformations of the associated superconformal field theories. In the case of extended $(2,2)$ supersymmetry, we discuss the transformations of the complex structures in relation to the superfield representation of the deformed theories. It is shown that generic deformations do not preserve the known superfield representations of the theories in terms of chiral and twisted chiral superfields. We also write down, explicitly, the transformations of the complex structures under the target space duality. The discussion clarifies the relation between the above duality transformation and a duality in the $N=2$ theories discussed by Gates, Hull and Roček [2]. We also comment on the possible form of the perturbative corrections to the $O(d,d)$ transformations in the context of $(4,4)$ theories.

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Note Added

When a complex structure depends on the coordinates with respect to which an $O(d,d)$ transformation is performed, then the extended supersymmetry in the deformed theory is no longer realized in the standard way. This issue was addressed in [20, 21] for the case of T-duality trasformations and it was shown that in the dual theory, the extended supersymmetry is realized non-locally. The generalization of this discussion to $O(d,d)$ transformations is straightforward.

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